Splitting the Gamma function

***Summary:** This paper shows that we can split the Gamma function in to two functions. The appendix gives details of the mathematical constants used in this paper together with a corollary.*

***Keywords:** Exponential function (e), Gamma (γ), Gamma function (Γ), Splitting.*

1. Splitting the Gamma function

We consider the following function to prove the results of splitting the Gamma function.

For real *r*, -∞ < *r* < +∞, let

$$f(r+2) = 1 + r(r+1)e^{-1}Y(r) + r(r+1)\sum_{m=2}^{\infty}\frac{(-1)^m}{(r+m)*m!},$$

with

$$Y(r+1) = rY(r) + 1$$

and

$$E(r) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(r+m)*m!}.$$

Then it is easy to show that

$$f(r+3) = (r+2)*\ f(r+2) \text{ with } f(1) = f(2) = 1.$$

2. Proof

$$f(r+3) = 1 + (r+1)(r+2)e^{-1}Y(r+1) + (r+1)(r+2)\sum_{m=2}^{\infty}\frac{(-1)^m}{(r+1+m)*m!}$$

and

$$(r+2)f(r+2) = (r+2)(1 + r(r+1)e^{-1}Y(r) + r(r+1)\sum_{m=2}^{\infty}\frac{(-1)^m}{(r+m)*m!}).$$

We now consider

$$f(r+2)*\ f(r+2) - f(r+3)$$

2.1 Consider the terms involving the terms involving *Y*(*r*). These are

$$(r+2)\left(r(r+1)e^{-1}Y(r)\right) - (r+1)(r+2)e^{-1}Y(r+1)$$
$$= (r+2)e^{-1}\left(r(r+1)Y(r)\right) - (r+1)(rY(r)+1))$$
$$= -(r+1)(r+2)e^{-1}.$$

2.2 Consider the terms involving a summation.

Note:

$$\sum_{m=2}^{\infty}\frac{(-1)^m}{(r+1+m)*m!}) = \sum_{t=3}^{\infty}\frac{(-1)^{t-1}}{(r+t)*(t-1)!} = \sum_{t=3}^{\infty}\frac{(-1)^{t-1}*t}{(r+t)*t!}$$

$$= \sum_{m=2}^{\infty}\frac{(-1)^{m-1}*m}{(r+m)*m!} + \frac{1}{(r+2)}.$$

Then the terms, involving the summations, are

$$(r+2)r(r+1)\sum_{m=2}^{\infty}\frac{(-1)^m}{(r+m)*m!}) - \ (r+1)(r+2)\sum_{m=2}^{\infty}\frac{(-1)^m}{(r+1+m)*m!}$$

$$= (r+2)(r+1)\sum_{m=2}^{\infty}\frac{r*(-1)^m}{(r+m)*m!}) + \ (r+1)(r+2)\sum_{m=2}^{\infty}\frac{m*(-1)^m}{(r+m)*m!} - (r+1).$$

$$= (r+2)(r+1)(\sum_{m=2}^{\infty}\frac{(r+m)*(-1)^m}{(r+m)*m!}) - (r+1) = (r+2)(r+1)e^{-1} - (r+1).$$

2.3 The remaining terms are: $(r+2)-1=(r+1)$.

2.4 Hence on adding the terms together the sum is

$$-(r+1)(r+2)e^{-1} + (r+2)(r+1)\ e^{-1} - (r+1) + (r+1) \equiv 0.$$

Hence

$$f(r+3) = (r+2)*\ f(r+2).$$

Now it is easy to show that

$$f(1) = f(2) = 1,$$

This all implies that *f*(*r*) is the Gamma function Γ(*r*). Therefore,

$$\Gamma(r+2) = 1 + \ r(r+1)e^{-1}Y(r) + r(r+1)\sum_{m=2}^{\infty}\frac{(-1)^m}{(r+m)*m!}\ . \qquad (1)$$

End of proof.

Hence, on dividing throughout by *r(*r*+1), we have that**

$$\Gamma(r) = e^{-1}Y(r) + \frac{1}{r(r+1)} + \sum_{m=2}^{\infty}\frac{(-1)^m}{(r+m)*m!}.$$

The last two terms sum to

$$E(r) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(r+m)*m!}.$$

Hence, in summary,

$$\Gamma(r) = e^{-1}Y(r) + E(r), \qquad (2)$$

where *Y*(*r*) is such that

$$Y(r+1) = rY(r) + 1$$

and

$$E(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(r+m) * m!}.$$

3. We can express (1) as

$$\frac{\Gamma(r+2) - 1}{r(r+1)} = e^{-1}Y(r) + \sum_{m=2}^{\infty} \frac{(-1)^m}{(r+m) * m!} .$$

It follows that for *r*=0, since

$$\Gamma'(2) = 1 - \gamma,$$

(Wikipedia, Gamma function) that

$$1 - \gamma = e^{-1}Y(0) + \sum_{m=2}^{\infty} \frac{(-1)^m}{m * m!} .$$

Hence

$$Y(0) = e * (\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m * m!} - \gamma) \qquad (3).$$

where γ ≈0.57721 56649 is the Euler–Mascheroni constant.

The summation in (3) is finite and is, approximately, 0.79659 95993 and, therefore, unlike $E(0)$ and $\Gamma(0), Y(0)$ is finite. Additionally, since for any integer *r* ≥ 1,

$$Y(-r) = \frac{Y(-r+1) - 1}{-r}$$

for any integer, *r*, *Y*(-*r*) is also finite. For all real *r*, except 0 and negative integers, both $E(r)$ and $\Gamma(r)$ are finite; it follows that, unlike $E(r)$ and $\Gamma(r)$, $Y(r)$ is continuous and that we can express

$$Y(r) = \sum_{j=0}^{\infty} y_j r^j$$

with

$$y_0 = Y(0) \approx 0.59634\ 73623.$$

<u>Reference</u>

Wikipedia (2022), 'Gamma function' accessed 26th. May 2022.

Appendix: Mathematical constants used in this note together with a corollary

1. Those accessed through Wikipedia

For a positive integer *r*, the derivative of the Gamma function can be calculated as follows:

$$\Gamma'(r+1) = r!\left(-\gamma + \sum_{k=1}^{r} \frac{1}{k}\right).$$

Here γ is the Euler–Mascheroni constant ≈ 0.57721 56649.

We also use the Euler constant *e* ≈ 2.71828 18284 and its inverse e^{-1}.

2. Two mathematical constants produced in this note and a useful corollary.

A. The following summation is used, in B, below

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m * m!}.$$

This is approximately: 0.79659 95993.

B. We produce the value of *Y*(0); and show that this is also used in the derivatives of the function *Y*(*r*).

$$y_0 = Y(0) = e * (\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m * m!} - \gamma).$$

Since *Y*(*r*+1) = *rY*(*r*) + 1, *Y'*(*r*+1) = *rY'*(*r*) + *Y*(*r*).

Corollary

Therefore, putting *r*=0, *Y'*(1) = *Y*(0) = y_0. That is,

$$Y'(1) = y_0 \approx 0.59634\,73623$$

Also putting *r*=1, *Y'*(2) = *Y'*(1) + *Y*(1) = y_0 + 1. It follows that, for positive integers, *r*, *Y'*(*r*), can be expressed in the form (*r*-1)!*y_0 + *G*(*r*) where *G*(*r*) is an integer for all positive integers *r*.

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